

22a) separable diff. eq.

$$y' = \frac{1-y^2}{1+x} \Rightarrow \frac{dy}{1-y^2} = \frac{dx}{1+x} \quad (2P)$$

$$\Rightarrow \int \frac{1}{2} \left(\frac{1}{1-y} + \frac{1}{1+y} \right) dy = \int \frac{1}{1+x} dx$$

$$\Rightarrow \frac{1}{2} \ln \left| \frac{1+y}{1-y} \right| = \ln |1+x| + C_1$$

$$\Rightarrow \left| \frac{1+y}{1-y} \right| = e^C (1+x) \Rightarrow \frac{1+y}{1-y} = D(1+x)^2 \quad (2P)$$

$$y(0) = 0 \Rightarrow \frac{1+0}{1-0} = D(1+0) = 0 \Rightarrow D = 1 \quad (2P)$$

$$\Rightarrow y = \frac{x^2 + 2x}{x^2 + 2x + 2} \quad (2P)$$

b) linear inhomogeneous $y' = \tan(x) \cdot y - x \quad (2P)$

integrating factor $u(x) = e^{\int \tan(x) dx} = \cos(x)$

$$\Rightarrow \frac{d}{dx} (\cos(x) y) = -x \cos(x)$$

$$\Rightarrow \cos(x) y = -\int x \cos(x) dx = -x \sin(x) - \cos(x) + C_1 \quad (2P)$$

$$y(0) = 2 \Rightarrow 2 = -1 + C_1 \Rightarrow C = 3 \quad (2P)$$

$$\Rightarrow y = -x \tan(x) - 1 + \frac{3}{\cos(x)} \quad (2P)$$

c) exact diff. eq

$$\frac{\partial (1+x)y}{\partial x} = 2y = \frac{\partial (1+2x+y^2)}{\partial y} \quad (2P)$$

$$(i.e. \frac{\partial F}{\partial x} = 1+2x+y^2, \frac{\partial F}{\partial y} = 2(1+x)y)$$

$$F = \int (1+2x+y^2) dx = x + x^2 + x y^2 + k(y)$$

$$\frac{\partial F}{\partial y} = 2(1+x)y = 2xy + k'(y) \Rightarrow k'(y) = 2y$$

$$\Rightarrow k(y) = y^2 + C_1$$

$$\Rightarrow F = x + x^2 + x y^2 + C_1 = (1+x)(y^2 + x) + C_1$$

$$\text{solution } (1+x)(y^2 + x) = D \quad (2P)$$

$$y(0) = -1 \Rightarrow 1(1+0) = D \Rightarrow D = 1 \quad (2P)$$

$$y = -\frac{1}{\sqrt{\frac{1}{1+x} - x}} = -\sqrt{\frac{1-x-x^2}{1+x}} \quad (2P)$$

d) linear homogeneous (second order)

$$9\lambda^2 + 12\lambda + 4 = (3\lambda + 2)^2 = 0$$

$$\Rightarrow \lambda^{(A)} = -\frac{2}{3} \quad (\text{repeated}) \quad (2P)$$

$$\Rightarrow y = C^{(A)} e^{-\frac{2}{3}x} + C^{(B)} x e^{-\frac{2}{3}x} \quad (2P)$$

$$y' = -\frac{2}{3} C^{(A)} e^{-\frac{2}{3}x} + C^{(B)} e^{-\frac{2}{3}x} - \frac{2}{3} C^{(B)} x e^{-\frac{2}{3}x}$$

$$\Rightarrow 1 = \gamma(0) = C^{(A)} \quad (2P) \Rightarrow 0 \quad C^{(B)} = \frac{2}{3} \quad (2P)$$

e) linear inhomogeneous (second order)

$$16\lambda^2 + 8\lambda + 5 = ((\lambda + 1)^2 + 4) = 0$$

$$\Rightarrow \lambda^{(A)} = -\frac{1}{4} + \frac{1}{2}i, \lambda^{(B)} = -\frac{1}{4} - \frac{1}{2}i \quad (2P)$$

$$\gamma P = A + Bx \Rightarrow 11 \cdot 0 + 8B + 5A + 5Bx = 10x$$

$$\Rightarrow A = -\frac{16}{5} \quad B = 2$$

$$\gamma = C^{(A)} e^{(-\frac{1}{4} + \frac{1}{2}i)x} + C^{(B)} e^{(-\frac{1}{4} - \frac{1}{2}i)x} + \frac{16}{5} + 2x \quad (2P)$$

$$-2 = \gamma(0) = C^{(A)} + C^{(B)} - \frac{16}{5} \quad (2P)$$

$$1 = (-\frac{1}{4} + \frac{1}{2}i) C^{(A)} + (-\frac{1}{4} - \frac{1}{2}i) C^{(B)} + 2$$

$$\Rightarrow C^{(A)} + C^{(B)} = \frac{6}{5} \Rightarrow C^{(A)} = \frac{3}{5} + \frac{7}{10}i, C^{(B)} = \frac{3}{5} - \frac{7}{10}i \quad (2P)$$

$$23a) \quad 0 = \det \begin{pmatrix} -3-\lambda & 1 \\ -2 & 1-\lambda \end{pmatrix} = \lambda^2 + 2\lambda - 1 = (\lambda+1)^2 - 2$$

$$\Rightarrow \lambda^{(A)} = -1 + \sqrt{2} \quad \lambda^{(B)} = -1 - \sqrt{2} \quad (4P)$$

$$\underline{y}^{(A)}: \begin{pmatrix} -3+1-\sqrt{2} & 1 \\ -2 & 1+1-\sqrt{2} \end{pmatrix} \underline{y}^{(A)} = 0 \Rightarrow \underline{y}^A = \begin{pmatrix} 1 \\ 2+\sqrt{2} \end{pmatrix}$$

(cos multiples) (2P)

$$\underline{y}^{(B)}: \begin{pmatrix} -3+1+\sqrt{2} & 1 \\ -2 & 1+1+\sqrt{2} \end{pmatrix} \underline{y}^{(B)} = 0 \Rightarrow \underline{y}^{(B)} = \begin{pmatrix} 1 \\ 2-\sqrt{2} \end{pmatrix}$$

(cos multiples) (2P)

$$b) \quad \lambda^2 + 3\lambda - 4 = (\lambda+4)(\lambda-1) = 0$$

$$\lambda^{(A)} = -4 \quad \lambda^{(B)} = 1 \quad (4P)$$

$$\underline{y}^{(A)}: \begin{pmatrix} 6+4 & 10 \\ -5 & -9+4 \end{pmatrix} \underline{y}^{(A)} = 0 \Rightarrow \underline{y}^{(A)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (2P)$$

(cos multiples)

$$\underline{y}^{(B)}: \begin{pmatrix} 6-1 & 10 \\ -5 & -9-1 \end{pmatrix} \underline{y}^{(B)} = 0 \Rightarrow \underline{y}^{(B)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (2P)$$

(cos multiples)

$$c) \quad \lambda^2 - 4\lambda + 5 = (\lambda-2)^2 + 1 = 0$$

$$\Rightarrow \lambda^{(A)} = 2+i, \lambda^{(B)} = 2-i \quad (= \lambda^{(A)*}) \quad (4P)$$

$$\underline{y}^{(A)}: \begin{pmatrix} 3-2-i & 1 \\ -2 & 1-2-i \end{pmatrix} \underline{y}^{(A)} = 0 \Rightarrow \underline{y}^{(A)} = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} \quad (2P)$$

(cos multiples)

$$\underline{y}^{(B)} = (\underline{y}^{(A)})^* = \begin{pmatrix} 1 \\ -1-i \end{pmatrix} \quad (\text{cos multiples}) \quad (2P)$$

$$d) \quad \lambda^2 - 1 = 0 \Rightarrow \lambda^{(A)} = 1, \lambda^{(B)} = -1 \quad (4P)$$

$$\underline{y}^{(A)}: \begin{pmatrix} 3-1 & -2 \\ 4 & -3-1 \end{pmatrix} \underline{y}^{(A)} = 0 \Rightarrow \underline{y}^{(A)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2P)$$

(cos multiples)

$$U^{(B)}: \begin{pmatrix} 3+i & -2 \\ 4 & -3+i \end{pmatrix} U^{(B)} = 0 \Rightarrow U^{(B)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (2P) \\ \text{(or multiples)}$$

$$e) \lambda^2 + 6\lambda + 25 = (\lambda + 3)^2 + 16 = 0 \quad (4P) \\ \Rightarrow \lambda^{(A)} = -3 + 4i, \lambda^{(B)} = -3 - 4i$$

$$U^{(A)}: \begin{pmatrix} -1+3-4i & 4 \\ -5 & -5+3-4i \end{pmatrix} U^{(A)} = 0 \Rightarrow U^{(A)} = \begin{pmatrix} 2 \\ -1+2i \end{pmatrix} \quad (2P) \\ \text{(or multiples)}$$

$$U^{(B)} = (U^{(A)})^* = \begin{pmatrix} 2 \\ -1-2i \end{pmatrix} \quad (2P)$$

$$24.) A \text{ and } III), \lambda = 2 \quad (2P) \text{ and/or } A \text{ and } IV), \lambda = -3 \quad (2P)$$

$$\text{(since } \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 3\pi \\ 2\pi \end{pmatrix} = 2 \begin{pmatrix} 3\pi \\ 2\pi \end{pmatrix})$$

$$\text{and/or } \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$B) \text{ and } V) \quad (2P) \quad \lambda = \sqrt{2} + i \quad (2P)$$

$$\text{(since } \begin{pmatrix} \sqrt{2} & 3 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -\sqrt{2}+i \end{pmatrix} = \begin{pmatrix} 3\sqrt{2}+3i \\ -\sqrt{2}+i \end{pmatrix} = (\sqrt{2}+i) \begin{pmatrix} 3 \\ -\sqrt{2}+i \end{pmatrix})$$

$$C) \text{ and } II \quad (2P) \quad \lambda = \sqrt{2} \quad (2P)$$

$$\text{(since } \begin{pmatrix} 2 & -1+\frac{1}{\sqrt{2}} \\ \sqrt{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \sqrt{2} \begin{pmatrix} 1 \\ 2 \end{pmatrix})$$

$$D) \text{ and } I) \quad (2P) \quad \lambda = i\sqrt{3} \quad (2P) \\ \text{(since } \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2\sqrt{3} \\ \sqrt{3}-3i \end{pmatrix} = i\sqrt{3} \begin{pmatrix} 2\sqrt{3} \\ \sqrt{3}-3i \end{pmatrix})$$

$$E) \text{ and } IV) \quad (2P) \quad \lambda = 1+i\sqrt{2} \quad (2P)$$

$$\text{(since } \begin{pmatrix} 1+i\sqrt{2} & \sqrt{2} \\ 2-\sqrt{2} & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (1+i\sqrt{2}) \begin{pmatrix} 1 \\ -1 \end{pmatrix})$$

H) Eigen values and eigenvectors of $\hat{A}$

$$\lambda^{(A)} = 1 \quad U^{(A)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad U^{(B)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ \lambda^{(B)} = 2$$

$$\hat{A} U^{(A/B)} = \lambda^{(A/B)} U^{(A/B)} = 0 \hat{A}^n U^{(A/B)} = (\lambda^{(A/B)})^n U^{(A/B)} \quad (10P) \quad (n=1,2,3,\dots)$$

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = U^{(A)} + U^{(B)}$$

$$\Rightarrow \hat{A}^{16} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \hat{A}^{16} U^{(A)} + \hat{A}^{16} U^{(B)}$$

$$= (\lambda^{(A)})^{16} U^{(A)} + (\lambda^{(B)})^{16} U^{(B)}$$

$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2^{16} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 65537 \\ 131071 \end{pmatrix} \quad (10P)$$